Anna Marciniak-Czochra (Heidelberg University)

Title: Mathematics of pattern formation: Emergence and stability of spatio-temporal structures

Abstract: The lectures are devoted to mathematical analysis of pattern formation models local and nonlocal interactions. Such models arise from applications in biosciences and describe coupling of diffusive or mechanical signalling and nonlinear intracellular feedbacks. We focus on two-component reaction-diffusion and reaction-diffusion-ODE systems which serve as basic models to understand pattern formation mechanisms. The presented theory includes the classical Turing-type pattern formation (based on diffusion-driven instability of spatially homogenous steady states), and more recent results based on existence of multi-stability and hysteresis. Comparing both mechanisms, we show that in reaction-diffusion-ODE models all close-to-equilibrium (Turing) patterns are unstable and only hysteresis-driven spatially heterogenous structures with jump discontinuities (far-from-equilibrium patterns) may be stable. The latter analysis requires new criteria for linear and nonlinear stability analysis. We characterize the spectrum of the linearized operator and relate its spectral properties to the corresponding semigroup properties. The applied methods of model analysis involve two-point boundary value problems, semigroup theory, spectral analysis and singular perturbation methods. Finally, we show that in some cases diffusion may lead to unbounded growth of solutions and mass concentration. The established mathematical theory is presented in the context of new models and experiments of symmetry breaking and pattern formation in Hydra, which is a model organism of developmental biology.

Sara Merino-Aceituno (University of Vienna)

Title: Pattern formation in models for collective motion using linear stability analysis

Abstract: Agent-based models for many-particles systems are typically hard to analyze mathematically. In contrast, the continuum descriptions of these systems are more prone to mathematical analyses, in particular, linear stability analysis. I will present how to apply linear stability analysis to investigate pattern formation in models for collective dynamics.

Ayman Moussa (Sorbonne Université)

Title: Cross Diffusion systems

Abstract: The purpose of this course is to study a class of systems of partial differential equations (PDE) used in population dynamics to describe the occupation of a space by different animal species. In a classical way, populations are described through two fundamental mechanisms: the dispersion of individuals (modeled by a diffusion operator) and their reproduction or death (modeled by a reaction term). The specificity of the systems on which we will focus lies in the expression "cross-diffusion": for such systems the diffusivity (or the motility) of a species depends - potentially in a non-linear way - on the presence of its competitors.

The first publication proposing such a system dates from 1979, in the Journal of Theoretical Biology. The authors, Shigesada, Kawasaki and Teramoto proposed this type of system (henceforth called "SKT") to capture segregation phenomena, that is: an almost disjoint distribution of space between the different constituents of the population. It is often the case in applied mathematics that an efficient modeling tool leads to interesting and surprisingly difficult mathematical questions; we will see in this course that cross-diffusion systems are a nice illustration of this fact.

After a quick introduction that will (formally) unveil the link between cross-diffusion and segregation, the course will first focus on the so-called Kolmogorov equation, a parabolic PDE whose diffusion operator is adapted to the behavior of sensitive individuals (as opposed to Fick's law, for the diffusion of lifeless matter). This equation being the basic building block of the cross-diffusion systems that we will study, it will be necessary to understand it in a framework of very weak regularity.

We will then approach the study of cross-diffusion systems. As it is often the case for nonlinear PDEs, we will see that the very question of the existence of solutions is not a triviality. We will provide a scheme for the construction of global weak solutions by approximation-compactness, based on the dissipation over time of an entropy functional and the persistence of this structure for non-local versions of the system. We will also explain how this existence proof gives in fact, in several cases, a rigorous justification of the model. If we have enough time, another derivation will be provided, based on a semi-discrete scheme.

In addition to exploring cross-diffusion systems, this course will illustrate some standard methods in the study of parabolic equations (maximum principle, Aubin-Lions lemma, compactness-approximation, fixed point in infinite dimension, asymptotic analysis) that we will present in the specific framework that interests us while underlining the wider scope of these tools.

Mariya Ptashnyk (Heriot-Watt University)

First lecture:

Title: Hopf Bifurcation for a system of parabolic equations with space-dependent coefficients

Abstract: We shall consider a mathematical model for a canonical gene regulatory network. The model consists of two partial differential equations describing the spatio-temporal interactions between the protein and its mRNA. Such intracellular negative feedback systems are known to exhibit oscillatory behaviour. An important feature of the model is that it has only space-dependent stationary solutions. Using the linearized stability analysis we shall show that the diffusion coefficient acts as a bifurcation parameter and gives rise to a Hopf bifurcation.

Second lecture:

Title: Chemotaxis equations and stochastic homogenization

Abstract: We will consider the Keller–Segel chemotaxis system in a random heterogeneous domain. First, we will analyse the system and derive a priori estimates that rely only on the boundedness of the coefficients; in particular, no differentiability assumption on the diffusion and chemotaxis coefficients for the chemotactic species is required. Then, we shall assume that the diffusion and chemotaxis coefficients are given by stationary ergodic random fields and apply stochastic two-scale convergence methods to derive the homogenized macroscopic equations.

Vivi Rottschäfer (University of Leiden)

Title: Pattern formation in reaction-diffusion systems

Abstract: In this lecture series, we consider and analyse patterns that arise when basic, steady state solutions become unstable. Moreover, we assume that this instability appears because of the presence of the diffusion terms. The patterns that occur through this mechanism are small amplitude solutions that oscillate around the steady state.

First, I will give an introduction to the general analysis of this phenomenon in reaction-diffusion systems. There I introduce the so-called Turing bifurcation. This Turing bifurcation occurs in many systems and equations, not only in those of reaction-diffusion type. Its presence and the implications of this has been widely studied.

The Turing bifurcation results in periodic solutions. I will give a derivation of an amplitude equation that describes the amplitude of such periodic solutions, the so-called Ginzburg-Landau equation.

Next, I will apply the linear stability of a solution of systems where mass is conserved. Systems are called conserved if neither mass is created nor destroyed within the system. These type of systems are abundant in nature. An example is a system of grazers like cows. In this system there exist two types of cows: grazing cows and moving cows. The system is conserved because the number of cows does not change. The herbivores prefer to graze in places with low vegetation, and therefore, they aggregate towards these places.

For a general mass conserved system, I will present the analysis leading to a Turing bifurcation. This analysis differs from that of the normal, non-conserved case. Also, I will show patterns that arise because of the presence of the Turing bifurcation. Apart from this, I will also briefly consider other patterns that arise far from equilibrium.